1-1-1966

Stability studies of automatic control systems.

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STABILITY STUDIES OF AUTOMATIC

CONTROL SYSTEMS

by

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A Thesis
Submitted to the Faculty of Graduate Studies through the
Department of Electrical Engineering in Partial Fulfillment
of the Requirements for the Degree of
Master of Applied Science at the
University of Windsor

Windsor, Ontario

1966

ABN8566

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ABSTRACT

This thesis presents a new and effective method for the analysis and synthesis of linear feedback control systems. The proposed method gives results comparable to those obtained by the conventional Nyquist and Root Locus methods and yields informations not obtainable from the other two methods.

By mapping the stability boundary of the complex s-plane onto the imaginary axis of the complex z-plane, Hurwitz criterion can be applied to yield analytical expressions, from which stability boundaries corresponding to different relative stability constraints can be drawn (α and ξ diagrams) for different values of system parameters. The stable and unstable regions are located and system analysis is reduced to the simple process of reading the diagrams. By superposition of the α and ξ diagrams, the dominant roots can be located which can be used for design purposes. The proposed method is most general and can be applied to systems with any degree of complexity and under any particular relative stability constraint. It has definite advantage over the popular Nyquist and Root Locus methods in many applications especially in the analysis and synthesis of systems with two or more variable parameters.

ACKNOWLEDGEMENT

The author wishes to express his sincere appreciation and gratitude to Dr. H. H. Hwang, who supervised this work, for his continuing encouragement and helpful advice.

Acknowledgement is also due to The National Research Council of Canada for the award of a scholarship which finances this project.

CONTENTS

		Page
ABSTRACT	....	iii
ACKNOWLEDGEMENT	....	iv
CONTENTS	....	v
CHAPTER		
I. INTRODUCTION	....	1
(a) Importance of Stability Studies		
(b) Classification of Stability		
(c) Conventional Methods for Relative Stability Studies		
II. A NEW ANALYTICAL APPROACH TO RELATIVE STABILITY STUDIES	....	4
(a) Formulation of The Hurwitz Criterion		
(b) Extension of Hurwitz Criterion to Relative Stability Studies		
III. APPLICATION TO PHYSICAL SYSTEM	....	12
(a) Characteristic Equation of Given System		
(b) α - Diagrams		
(c) ζ - Diagrams		
(d) Location of Stability Regions		
(e) Location of Dominant Roots		
(f) Interpretation of Diagrams		
(g) System Analysis and Design		
IV. DISCUSSION AND CONCLUSION	....	26

APPENDIX 27

(1) TABLE FOR COMPUTING COEFFICIENTS OF F(Z)

(2) α - DIAGRAMS

(3) ξ - DIAGRAMS

(4) SUPERPOSITION OF α AND ξ DIAGRAMS

REFERENCES 38

VITA AUCTORIS 39

CHAPTER I

INTRODUCTION

Importance of Stability Studies

Stability studies is one of the most important topies in feedback control systems. From a classical point of view, a central problem of feedback control theory can be identified as a stability problem. The purpose of any feedback control system is to have the controlled output of the system bear a definite and known relationship to the reference input. Therefore it is imperative that the transient should die down after the cessation of the disturbance. Systems in which the transient increases without bound after cessation of the disturbance are said to be unstable. Instability is undesirable in that the controlled output is not under control and it may cause harm and failure of the system. Therefore it is most important to have an effective method whereby the stability problem can be readily studied.

Classification of Stability

In general, stability can be studied under two different headings namely, (1) Absolute Stability and (2) Relative Stability. A system is said to be absolutely stable if the transient dies down as time approaches infinity, and a system is said to be relatively stable if the transient dies down within a certain period of time or a specified number of cycles. For any feedback control system, absolute stability is a necessary condition. But it is not sufficient because a system may be absolutely stable, but if the transient takes such a long time to decay that it is not practical for use. Therefore, relative stability is a more important requirement for a system than absolute stability because once it is established that a system is relatively stable, it is automatically absolutely stable.

The problem of absolute stability is well understood and many methods have been developed by which the absolute stability of a system can be readily determined. However, the measurement of relative stability has not been developed to the state where comparable definite techniques are available. The quantities are not well defined and the interpretation of relative stability differs among individuals. In general, there are two criteria which are commonly used to specify the relative stability requirements of a system. One of these is the requirement that all the transient terms of the response must decay at least as rapidly as the function $e^{-\alpha t}$. This means that all the roots of the characteristic equation must be located on the left side of a line parallel to and at a distance $\sigma = -\alpha$ from the imaginary axis as shown in figure 1-1.

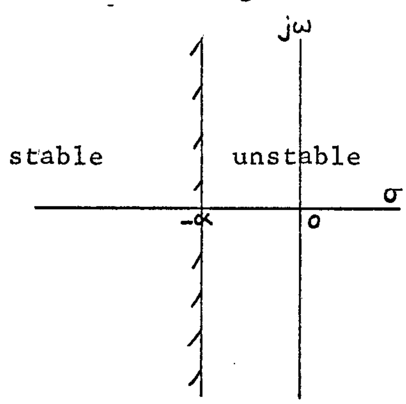

Fig. 1-1

S-PLANE

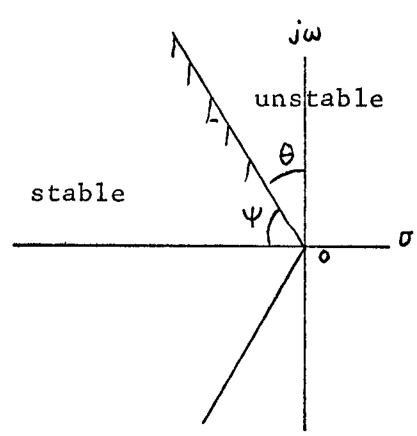

Fig. 1-2

The other is the requirement that all oscillatory terms in the transient response must die down in a specified number of cycles. This is equivalent to specifying a minimum value of the damping ratio ξ that can be tolerated. This requires that all the roots of the characteristic equation must be located within the sector bounded by the constant ξ line as shown in figure 1-2.

Conventional Methods for Relative Stability Studies

The most popular methods which are being used for relative stability studies are the Nyquist and Root Locus methods. The former is a frequency domain approach which permits the designer to modify the open loop system in order to obtain the appropriate closed loop frequency characteristics. However, it can be applied to systems where the open loop function is known within a gain factor. Each time a time constant or parameter is changed, a new Nyquist diagram must be drawn. Also if the transfer function is very complex and not in factored form, Nyquist plots may be very difficult. The latter is a pole and zero approach which readily provides importmations about all the roots of the characteristic equation for a given value of the open loop gain. However, it also has many significant limitations. The most important of which is the fact that it is basically a one parameter method which is very inconvenient for analysis and synthesis of multi-parameter systems. Therefore better and more effective methods are desirable. In the chapters following, a new analytical method is developed which can be applied to syslems of any degree of complexity. The method is especially suitable for the analysis and synthesis of systems with two or more variable parameters.

CHAPTER II

A NEW ANALYTICAL APPROACH TO RELATIVE STABILITY STUDIES

As mentioned in the previous chapter, the problem of relative stability is a problem of detecting whether or not any roots of the characteristic polynomial lie outside a specified stability region. Thus stability studies is reduced to the study of the location of the roots of the characteristic equation of the system.

The Hurwitz criterion has been one of the most popular tools in absolute stability analysis. When applied to a characteristic equation with real coefficients, the Hurwitz criterion readily indicates the existence of roots with positive real parts and thus indicating instability. However, it has the obvious limitation of being only able to be applied to equations with real coefficients. Also it does not indicate the degree of stability of the system and it gives little insight into system design. Thus it has had very little application in relative stability studies. However, with some manipulations, the Hurwitz criterion can be directly applied to yield information about relative stability. It also yields analytical expressions from which stability boundaries can be drawn. Thus information about the dynamical behaviour of the system can be obtained in a simple and straight forward manner.

Formulation of The Hurwitz Criterion

Hurwitz formulated his criterion in a determinantal form and stated it in the following way.[4] For a polynomial

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \text{------} + a_1 s + a_o$$
$$= \sum_{k=0}^{n} a_k s^k \qquad (2\text{-}1)$$

where a_k's are real, all the roots of the characteristic equation $P(s) = 0$ will have negative real parts if the following conditions are satisfied.

(1) $a_k > 0$

(2) the Hurwitz determinants $\Delta_k > 0$ for $k = 1, 2, \text{-----} n-1$

where

$$\Delta_k = \begin{vmatrix} a_1 & a_0 & 0 & 0 & \text{---------} & 0 \\ a_3 & a_2 & a_1 & a_0 & \text{---------} & 0 \\ a_5 & a_4 & a_3 & a_2 & \text{---------} & 0 \\ \text{------} & \text{------} & \text{------} & \text{------} & \text{------} & \text{------} \\ 0 & \text{------} & \text{------} & \text{------} & \text{------} & a_{n-1} \end{vmatrix} \qquad (2\text{-}2)$$

If some of the Hurwitz determinants are negative, this indicates the existence of roots with positive real parts and the number of such roots corresponds to the number of changes of sign in the sequence $\Delta_1 \Delta_2 \Delta_3 \text{----} \Delta_{n-1}$. If the characteristic equation has roots lying on the imaginary axis, then the n-1th Hurwitz determinant will vanish[3].

Extension of Hurwitz Criterion to Relative Stability Studies

It has been mentioned that the Hurwitz criterion, when applied to polynomials with real coefficients, readily indicates the existence of roots with positive real parts and thus indicates instability. Therefore the application of the Hurwitz criterion to relative stability studies generally consists of one or both of the following processes;

(1) the mapping of the stability boundary onto the imaginary axis of a new plane.

(2) the realization of the coefficients of the transformed polynomial.

For systems under the first relative stability constraint (minimum α), the requirement is that all the roots of the characteristic equation be located on the left side of a line parallel to and at a distance $\sigma = -\alpha$ from the imaginary axis of the s-plane as shown on figure 2-1(a).

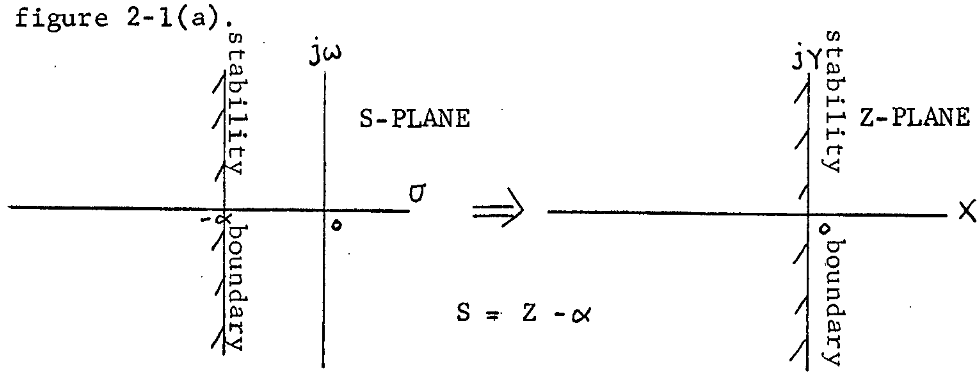

Fig. 2-1(a)

STABILITY BOUNDARY BEFORE TRANSFORMATION

Fig. 2-1(b)

STABILITY BOUNDARY AFTER TRANSFORMATION

let the characteristic polynomial be

$$P(s) = \sum_{k=0}^{n} a_k s^k \tag{2-1}$$

where a_k are real.

By using the transformation $s = Z-\alpha$, the $\sigma = -\alpha$ boundary of the complex s-plane is mapped onto the imaginary axis of the Complex Z-plane. The transformed polynomial is,

$$P(Z-\alpha) = \sum_{k=0}^{n} a_k (Z-\alpha)^k = \sum_{k=0}^{n} C_k Z^k \qquad (2\text{-}2)$$

where C_k's are coeffs. of $P(Z-\alpha)$

Since the a_k's are real and α is real, therefore the coefficients C_k's of the transformed polynomial are real. Hence, the Hurwitz criterion can be applied directly to $P(Z-\alpha)$ to yield informations about the dynamical behaviour of the system.

For systems under the second kind of relative stability constraint (minimum ξ), the requirement is that all the roots of the characteristic equation be located within the sector bounded by the minimum ξ lines (Fig. 2-2).

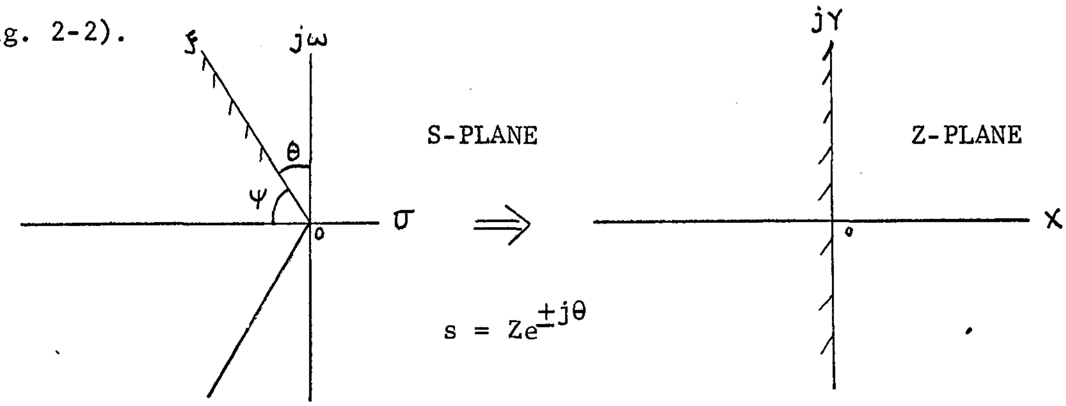

Fig. 2-2(a)

STABILITY BOUNDARY CORRESPONDING TO THE SECOND RELATIVE STABILITY CONSTRAINT

Fig. 2-2(b)

STABILITY BOUNDARY OF TRANSFORMED POLYNOMIAL

By using similar techniques, the stability boundary can be mapped onto the imaginary axis of the complex Z-plane as shown on figure 2-2 (b).

Consider the characteristic polynomial.

$$P(s) = \sum_{k=0}^{n} a_k s^k$$

$$= a_n s^n + a_{n-1} s^{n-1} + \text{------} + a_1 s + a_0 \qquad (2\text{-}1)$$

where a_k's are real.

let $\qquad s = Ze^{j\theta} \qquad (2\text{-}3)$

then $\qquad P_1(Z) = \sum_{k=0}^{n} a_k (Ze^{j\theta})^k$

$$= a_n e^{jn\theta} Z^n + a_{n-1} e^{j(n-1)\theta} Z^{n-1} + \text{----}$$

$$+ a_1 e^{j\theta} Z + a_0 \qquad (2\text{-}4)$$

Physically, the transformation $s = Ze^{j\theta}$ corresponds to a clockwise rotation of the roots of the characteristic polynomial P(s) through an angle of θ degrees as shown in figure 2-3(a)(b).

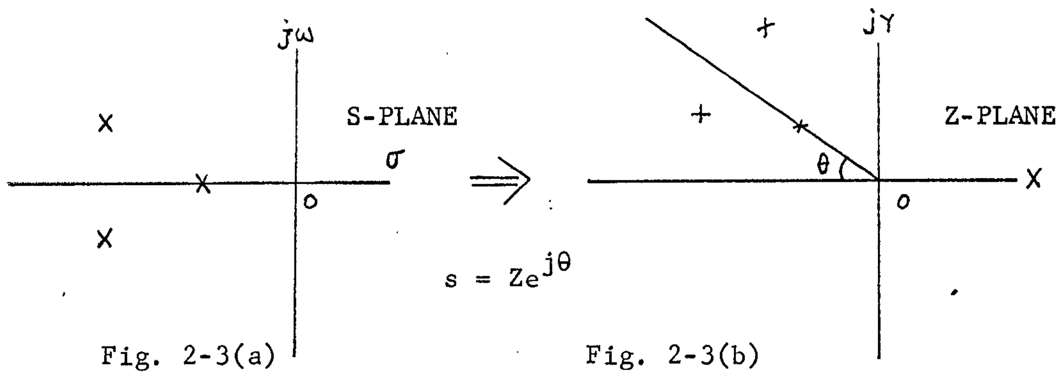

Fig. 2-3(a) LOCATION OF ZEROS OF P(s)

Fig. 2-3(b) LOCATION OF ZEROS OF $P_1(Z)$

Mathematically, the transformation $s = Ze^{j\theta}$ gives rise to a polynomial $P_1(Z)$ with complex coefficients[(6)] as shown in equation 2-4. Since the Hurwitz criterion is applicable only to polynomials with real coefficients, therefore further operations must be performed to realize the transformed polynomial. To do this, let

$$s = Ze^{-j\theta} \qquad (2\text{-}5)$$

and form $P_2(Z) = \sum_{k=0}^{n} a_k (Ze^{-j\theta})^k$

$$= a_n e^{-jn\theta} Z^n + a_{n-1} e^{-j(n-1)\theta} Z^{n-1} + \text{-----}$$
$$+ a_1 e^{-j\theta} Z + a_0 \qquad (2\text{-}6)$$

This corresponds to a counterclockwise rotation of the roots of P(s) through an angle of θ degrees as shown in figure 2-4(a)(b).

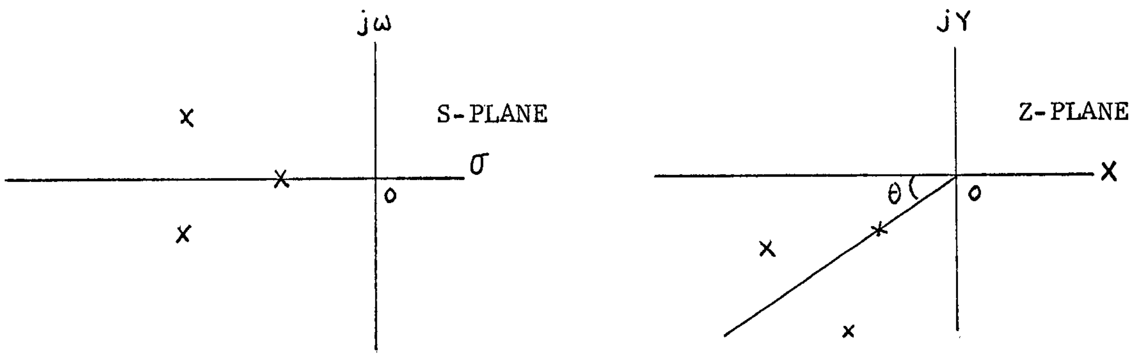

Fig. 2-4(a)

ZEROS OF P(s)

Fig. 2-4(b)

ZEROS OF $P_2(Z)$

By multiplying $P_1(Z)$ and $P_2(Z)$ together, a new polynomial is obtained.

$$F(Z) = P_1(Z)P_2(Z) = \sum_{k=0}^{n} a_k (Ze^{j\theta})^k \sum_{k=0}^{n} a_k (Ze^{-j\theta})^k$$
$$= \sum_{k=0}^{2n} A_k Z^k \qquad (2\text{-}6)$$

where A_k's are coeffs. of F(Z)

Physically, equation (2-6) corresponds to a superposition of the zeros of $P_1(Z)$ and $P_2(Z)$ as shown in figure 2-5.

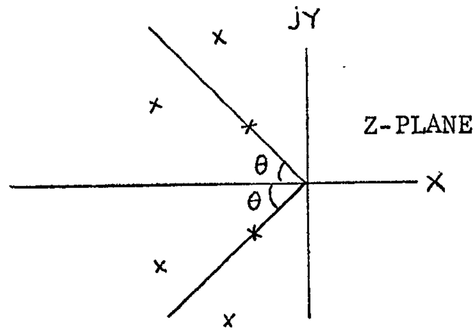

Fig. 2-5 ZEROS OF F(Z)

It can be seen that the zeros of F(Z) occur in complex conjugate pairs and therefore the coefficients of F(Z) are real coefficients. However, it is observed that F(Z) has twice as many roots as the original polynomial. This is because of the fact that in the process of realizing the coefficients, we introduced conjugate roots. But nevertheless, this has no effect on our analysis because in relative stability analysis, we are only interested in the location of the roots of the characteristic equation and not the number. To illustrate the above process, consider a second order system whose characteristic polynomial is given by,

$$P(s) = a_0 + a_1 s + a_2 s^2$$

then $$P_1(Z) = a_0 + a_1 e^{j\theta} Z + a_2 e^{j2\theta} Z^2$$

and $$P_2(Z) = a_0 + a_1 e^{-j\theta} Z + a_2 e^{-j2\theta} Z^2$$

$$\begin{aligned} F(Z) &= P_1(Z) P_2(Z) \\ &= (a_0 + a_1 e^{j\theta} Z + a_2 e^{j2\theta} Z^2)(a_0 + a_1 e^{-j\theta} Z + a_2 e^{-j2\theta} Z^2) \\ &= a_0^2 + (2a_0 a_1 \cos\theta) Z + (a_1^2 + 2a_0 a_2 \cos 2\theta) Z^2 \\ &\quad + (2a_1 a_2 \cos\theta) Z^3 + a_2^2 Z^4 \end{aligned} \tag{2-7}$$

$$F(Z) = \sum_{k=0}^{4} A_k Z^k \tag{2-8}$$

where A_k's are real coefficients given in equation 2-7.

Similarly for an n^{th} order characteristic polynomial, the transformed polynomial can be obtained by the same procedure. The coefficients A_k's can be computed once and for all and put in a table. Thus for any given characteristic polynomial P(s), the corresponding transformed polynomial F(Z) can be obtained readily by referring to the table. A table for computing the coefficients of a 10^{th} order polynomial has been workedout and is shown in Table I.

The above method is most general and can be applied to the relative stability studies of any system under any particular kind of relative stability constraint. By mapping the stability boundary onto the imaginary axis of a new plane, and realizing the coefficients of the transformed polynomial the Hurwitz criterion can be applied easily to yield informations about the dynamical behaviour of the system. Furthermore, by equating the $n-1^{th}$ Hurwitz determinant to zero, analytical expressions can be obtained from which stability boundaries can be drawn for different values of system parameters. Thus relative stability studies is further simplified to reading diagrams.

CHAPTER III

APPLICATION TO PHYSICAL SYSTEM

The versatility of the method can be best illustrated when applied to systems with two or three variable parameters. Consider a physical system[1] whose diagram is given as follows.

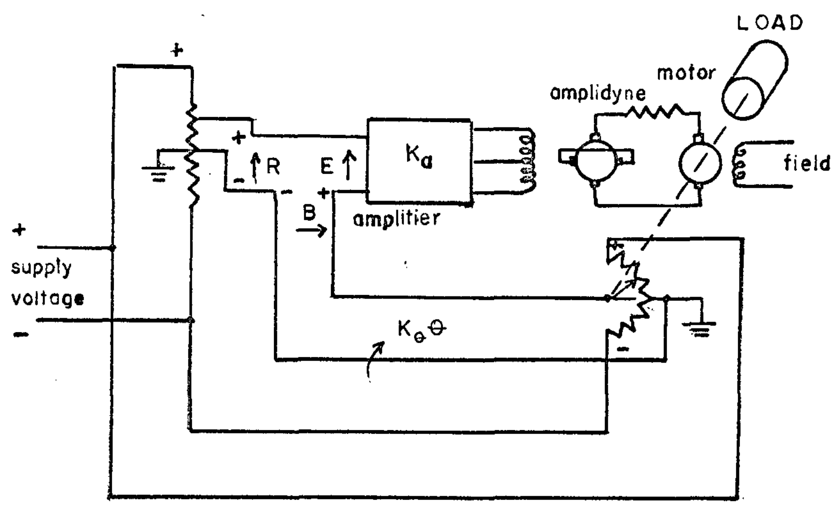

Fig. 3-1 TYPICAL POSITION CONTROL SYSTEM[1]

The system parameters are given by[1]:

Ka = net control field amperes per volt error signal

Kg = no-load amplidyne terminal voltage per net control field current,

$T_f = \frac{L_f}{R_f}$ = time constant of amplidyne quadrature field in seconds

$T_m = JRa/k_T \cdot K_e$ = time constant of motor and load in seconds

K_e = motor volts per radian per second of motor

K_T = torque from motor per ampere of motor armature current

K_θ = voltage from feedback potentiometer per radian of motor

R_a = total armature resistance of motor amplidyne and leads.

From the given system, the following block diagram is obtained.

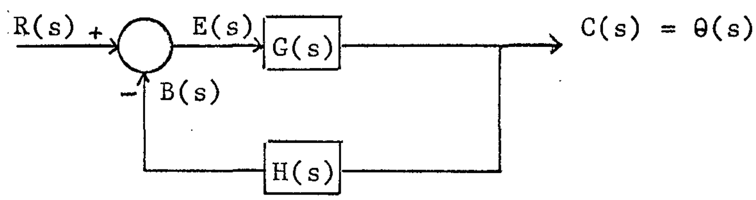

Fig. 3-2 BLOCK DIAGRAM

where

R(s) = reference input

E(s) = error or actuating signal

B(s) = feedback signal

G(s) = open loop transfer function

H(s) = feedback transfer function

C(s) = controlled output

In terms of E(s), the transform of the position of the motor $\theta(s)$ can be written as:(1)

$$\theta(s) = \frac{K_a K_g E(s)}{K_e\, s(1+T_f s)(1+T_m s)}$$

$$G(s) = \frac{\theta(s)}{E(s)} = \frac{K_a K_g/K_e}{s(1+T_f s)(1+T_m s)}$$

and $$H(s) = \frac{B(s)}{\theta(s)} = K_\theta$$

$$G(s)\,H(s) = \frac{K_aK_gK_\theta/K_e}{s(1+T_fs)(1+T_ms)}$$

$$\doteq \frac{K}{s(1+T_fs)(1+T_ms)}$$

where $K = K_aK_gK_\theta/K_e$

= overall system gain

The characteristic equation is $1+G(s)H(s) = 0$

or $$1 + \frac{K}{s(1+T_fs)(1+T_ms)} = 0$$

from which $P(s) = K + s + (T_f + T_m)s^2 + T_fT_ms^3$

$$= K + s + a_2s^2 + a_3s^3 \qquad (3\text{-}1)$$

where $a_2 = T_f + T_m$

$a_3 = T_fT_m$

α - Diagrams

The first criterion of relative stability requires all the roots of the characteristic equation to have negative real parts less than a specified value of $\sigma = -\alpha$. Using the transformation $s = Z-\alpha$, the stability boundary ($\sigma = -\alpha$ line) is mapped onto the imaginary axis of the complex Z-plane.

$$P(s) = K + s + a_2s^2 + a_3s^3 \qquad (3\text{-}1)$$

$$P(Z-\alpha) = K + (Z-\alpha) + a_2(Z-\alpha)^2 + a_3(Z-\alpha)^3$$

$$= a_3Z^3 + (a_2-3\alpha a_3)Z^2 + (1+3\alpha^2a_3-2\alpha a_2)Z$$

$$+ (K+\alpha^2a_2-\alpha-\alpha^3a_3) \qquad (3\text{-}2)$$

from (3-2), the Hurwitz determinant is:

$$\Delta_k = \begin{vmatrix} (1+3\alpha^2 a_3-2\alpha a_2) & (K+\alpha^2 a_2-\alpha-\alpha^3 a_3) & 0 \\ a_3 & (a_2-3\alpha a_3) & (1+3\alpha^2 a_3-2\alpha a_2) \\ 0 & 0 & a_3 \end{vmatrix} \quad (3\text{-}3)$$

The n-1th Hurwitz determinant will vanish if there are roots of the characteristic equation located on the imaginary Z axis which is the relative stability boundary. Hence, by equating the n-1th Hurwitz determinant to zero, an analytical expression can be obtained from which relative stability boundaries can be drawn[2,3]. From (3-2).

$$\Delta_{n-1} = \Delta_2 = \begin{vmatrix} 1+3\alpha^2 a_3-2\alpha a_2 & K+\alpha^2 a_2-\alpha-\alpha^3 a_3 \\ a_3 & a_3-3\alpha a_3 \end{vmatrix} \quad (3\text{-}4)$$

Equating $\Delta_{n-1} = 0$, get

$$(1+3\alpha^2 a_3-2\alpha a_2)(a_3-3\alpha a_3) - a_3(K+\alpha^2 a_2-\alpha-\alpha^3 a_3) = 0$$

Expanding and rearranging, get

$$K = \frac{1}{a_3}\left[\, 8\alpha^2 a_3(a_2-\alpha a_3) - 2\alpha(a_2^2+a_3) + a_2 \,\right] \quad (3\text{-}5)$$

equation for plotting relative stability boundaries (α - diagrams). Equation (3-5) is plotted for various values of α and system parameters as shown on Diagrams 1, 2 and 3 (α - diagram).

ξ - Diagrams

When the relative stability constraint is the minimum value of the damping ratio ξ, the relative stability region becomes the sector bounded by the constant ξ line as shown in Fig. 2-2(a). By using the transformation $s = Ze^{\pm j\theta}$ and Table I, the transformed polynomial F(Z) can be obtained from which the Hurwitz determinant Δ_k can be formed. By equating the n-1th Hurwitz determinant to zero, an analytical expression is obtained from which relative stability boundaries corresponding to a particular value of ξ can be drawn (ξ - diagrams). The minimum value of the damping ratio differs for different systems according to the particular requirement. But for ordinary systems, a damping ratio of 0.5 is usually adequate.

let $\xi = 0.5$

then $\psi = \cos^{-1}\xi = 60^{\circ}$

$\theta = 90^{\circ} - 60^{\circ} = 30^{\circ}$

The characteristic equation of the system under study is

$$P(s) = K + s + a_2 s^2 + a_3 s^3 \qquad (3\text{-}1)$$

The transformed polynomial F(Z) is written in general form as:

$$F(Z) = A_0 + A_1 Z + A_2 Z^2 + A_3 Z^3 + A_4 Z^4 + A_5 Z^5 + A_6 Z^6 \qquad (3\text{-}6)$$

where the coefficients are computed by referring to Table I,

$A_0 = a_0^2 = K^2$

$A_1 = 2a_1 a_0 \cos\theta = bK$ where: $b = 1.732$

$A_2 = a_1^2 + 2a_2 a_0 \cos 2\theta = 1 + Ka_2$ $a_2 = T_f + T_m$

$A_3 = ba_2$

$A_4 = a_2^2 + a_3$ $a_3 = T_f T_m$

$A_5 = ba_2 a_3$

$A_6 = a_3^2$

From equation 3-6:

$$\Delta_{n-1} = \Delta_5 = \begin{vmatrix} A_1 & A_0 & 0 & 0 & 0 \\ A_3 & A_2 & A_1 & A_0 & 0 \\ A_5 & A_4 & A_3 & A_2 & A_1 \\ 0 & A_6 & A_5 & A_4 & A_3 \\ 0 & 0 & 0 & A_6 & A_5 \end{vmatrix}$$

$$= \begin{vmatrix} bK & K^2 & 0 & 0 & 0 \\ ba_2 & 1+Ka_2 & bK & K^2 & 0 \\ ba_2a_3 & a_2^2+a_3 & ba_2 & 1+Ka_2 & bK \\ 0 & a_3^2 & ba_2a_3 & a_2^2+a_3 & ba_2 \\ 0 & 0 & 0 & a_3^2 & ba_2a_3 \end{vmatrix} \qquad (3\text{-}7)$$

Equating Δ_{n-1} to zero and simplifying, we obtain:

$$(a_2^3a_3^2)K^3 + (a_3^3 - a_2^2a_3^2 - 2a_2^4a_3)K^2 + (a_2^5 + 2a_2^3a_3 - 2a_2a_3^2)K + (a_2^2a_3 - a_2^4) = 0 \qquad (3\text{-}8)$$

From equation (3-8), stability boundaries are drawn for different values of system parameters as shown on diagrams 4, 5, and 6 (ξ - diagrams).

Location of Stability Regions on α and ξ Diagrams

(A) α - Diagrams

After the stability boundaries have been plotted, the next thing is to locate the stable and the unstable regions. This can be done sometimes by inspection. For complicated cases where the stability regions cannot be determined by inspection, the following method can be used.

(1) Select two points, one on each side of the stability boundary.

(2) Apply the Hurwitz criterion to test the two points.

(3) The point which satisfies the Hurwitz criterion lies on the stable region while the one, which does not, lies on the unstable region.

For example, referring to diagram 3, let the relative stability constraint be $\alpha = 1.0$. Then the stability boundary corresponding to this particular requirement is the $\alpha = 1.0$ curve. To locate the stability region, select two points one on each side of the $\alpha = 1.0$ boundary. Let the points be:

(1) Point A with $T_f = 0.4$, $T_m = 0.08$, $K = 1$, $a_2 = 0.48$, $a_3 = 0.032$

(2) Point B with $T_f = 0.4$, $T_m = 0.08$, $K = 3$

Applying Hurwitz criterion to point A, and from (3-3)

$$\Delta_k = \begin{vmatrix} 1+3(.032)-2(.48) & 1+.48-1-.032 & 0 \\ .032 & .48-3(.032) & .136 \\ 0 & 0 & .032 \end{vmatrix}$$

From which $\Delta_1 = .136$, $\Delta_2 = 0.0379$, and $\Delta_3 = 0.0012$. Since $\Delta_1 \Delta_2$ and Δ_3 are all positive, therefore point A lies on the stable region.

For point B.

$$\Delta_k = \begin{vmatrix} .136 & 2.448 & 0 \\ .032 & .384 & .136 \\ 0 & 0 & .032 \end{vmatrix}$$

From which

$$\triangle_1 = 0.136$$

$$\triangle_2 = -0.0262$$

$$\triangle_3 = -0.00084$$

Since $\triangle_1 \triangle_2$ and $\triangle_3$ have different signs, therefore point B lies on the unstable region. From the above two tests, we can conclude that the region on the left of the stability boundary corresponds to the stable region while the region on the right corresponds to the unstable region.

(B) ξ - Diagrams

The stable and unstable regions of the ξ - diagrams can be located by similar procedures. It is found that for the ξ - diagrams, the stable region also lies on the left of the relative stability boundary while the unstable region lies on the right. With the α and ξ diagrams available and the stability regions located, the dynamical behaviour of the system can be easily visualized, and relative stability analysis becomes a simple matter of reading the diagrams.

Location of Dominant Poles

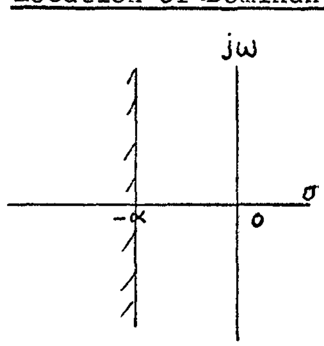

Fig. 3-3(a)

STABILITY BOUNDARY OF

α - DIAGRAMS

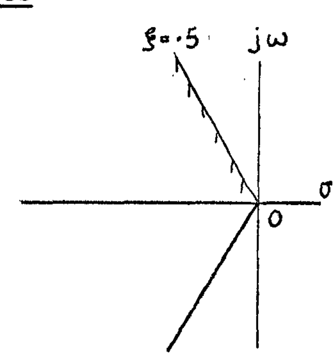

Fig. 3-3(b)

STABILITY BOUNDARY OF

ξ - DIAGRAMS

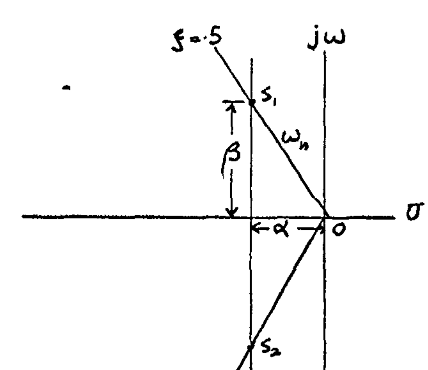

Fig. 3-3(c)

SUPERPOSITION OF

α AND ξ DIAGRAMS

The behaviour of a system is largely determined by the pair of roots located closest to the imaginary axis. For this reason, such roots are called dominant roots because they dominate the behaviour of the system. For a given relative stability requirement, the dominant roots can be located by a simple superposition of the α and ξ diagrams. Each α diagram corresponds to a stability boundary for a given value of α (Fig. 3-3(a)), and each ξ diagram corresponds to a stability boundary for a given value of ξ (fig. 3-3(b)). Therefore by superposing the α and ξ diagrams, the point of intersection between the two curves gives the location of the dominant poles which will satisfy the required relative stability constraints and therefore will give the desired system performance.

Interpretation of Diagrams

The α diagrams are relative stability boundaries for various values of α and system parameters when the magnitude of the real part of the roots is the relative stability constraint. For example, any point on the $\alpha = 1.0$ curve corresponds to a root of characteristic equation with negative real part $\alpha = 1.0$. Points lying on the left of the boundary correspond to roots with negative real part less than 1.0 while points on the right correspond to roots with negative real part greater than 1.0. Thus the region on the left represents the stable region and the one on the right represents the unstable region. The ξ diagrams are stability boundaries when the minimum value of the damping ratio is the stability constraint. Any point on the ξ diagram corresponds to a root of the characteristic equation with damping ratio $\xi = 0.5$. Points on the left of the boundary correspond to roots with damping ratio greater than the minimum requirement while points on the right correspond to roots with damping ratio less than the required minimum. Thus the stable region is located on the left while the unstable region is located on the right of the stability boundary.

From both the α and ξ diagrams, it is observed that for constant values of T_m, the stability boundaries bend towards the left as T_f increases. This means that an increase in T_f must be accompanied by a decrease in the overall gain to maintain system stability. When T_m is also allowed to increase, the stability boundaries bend more towards the left. This means that the system gain must be further reduced. Thus not only can we determine the dynamical behaviour of the system under any given conditions, but we can also visualize the effect of simultaneous variation of system parameters on the performance of the system. This information is particularly important especially in guided missile systems and other position control systems where the time constants are functions of temperature and other enviromental conditions.

From diagram 8, the superposition of α and ξ diagrams, it is observed that the $\xi = 0.5$ boundary does not cut the α durves for values of α less than 0.6. This means that the dominant roots cannot have real parts with magnitude less than 0.6 for the given range of values of system parameters. Thus this method places a limitation on the magnitude of the dominant roots for a given relative stability requirement.

System Analysis and Design

The α and ξ diagrams can be used most effectively for system analysis as well as design. The diagrams readily give information with regard to the relative stability of the system for any given values of system parameters. For example, for a system with parameters given by $T_f = 0.64$ sec., $T_m = 0.06$ sec. and $K = 1$, information about the dynamical behaviour of the system can be obtained from the α - diagram or the ξ diagram or the combination of both depending on the particular relative stability constraint. Let it be required that the characteristic equation must have roots with negative real parts less than -0.8 and damping ratio not less than 0.5. Referring to diagram 8, it is observed that the point $T_f = 0.64$ sec., $T_m = 0.06$ sec. and $K = 1$ (point c) falls on the left of the $\xi = 0.5$ boundary but on the right of the $\alpha = 0.8$ boundary. This means that the roots of the characteristic equation will have damping ratio greater than 0.5 but with negative real parts greater than -0.8. Thus the relative stability requirements are not satisfied and the system, under the above given conditions, is relatively unstable. Similarly, for other given system conditions and relative stability requirements, the dynamical behaviour of the system can be determined by simply referring to the diagrams.

The performance of a system can be approximated in terms of a second order system when the dominant poles are located. Referring to diagram 8, the $\xi = 0.5$ and $\alpha = 1.2$ curves intersect at the point where $T_f = 0.36$ sec., $T_m = 0.06$ sec. and $K = 2.05$ (Point D) from which the dominant poles can be located.

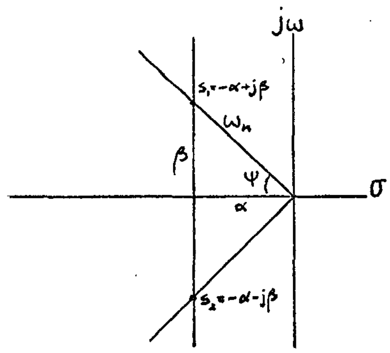

Fig. 3-4

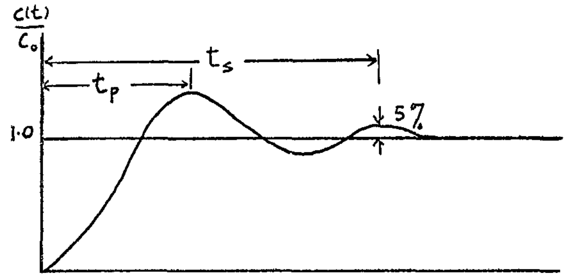

Fig. 3-5 UNIT STEP RESPONSE OF 2^{nd} ORDER SYSTEM

$\xi = 0.5 \qquad \theta = 60^{\circ}$

$\alpha = 1.2$

$\beta = \alpha \tan\theta = 1.2 \tan 60^{\circ} = 2.08$

the dominant roots occur at

$$s = -1.2 \pm j2.08$$

the natural frequency of the system is

$$W_n = \frac{\alpha}{\xi} \doteq \frac{1.2}{0.5} = 2.4 \quad \text{radions/sec.}$$

for a unit step input the bandwidth is given by[(1)]

$$W_b = W_n\sqrt{1 - 2\xi^2 + \sqrt{2 - 4\xi^2 + 4\xi^4}}$$

$$= 2.4 \times 1.27$$

$$= 3.05$$

the time for the controlled output to reach its maximum overshoot is:

$$t_p = \frac{\pi}{\sqrt{1-\xi^2}\, w_n} = \frac{\pi}{\beta} = \frac{3.14}{2.08} = 1.51 \text{ secs.}$$

The settling time is given by:

$$t_s = \frac{3.0}{w_n} = \frac{3.0}{0.5x2.4} = 2.5 \text{ secs.}$$

The maximum overshoot is:

$$\left.\frac{c(t)}{c_o}\right|_{t=t_p} = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} = 0.166\%$$

and the number of ocillations N for the system to settle is:

$$N = \frac{1.5\sqrt{1-\xi^2}}{\pi\xi} = 0.82$$

Thus complete information about system performance is obtained when the dominant roots are located. Furthermore the α and ξ diagrams can be used to obtain the values of system parameters that will give a required response. For example, if it be desired that the system should have a response with damping ratio $\xi = 0.5$ and resonant frequency $w_n = 2.2$ radians/sec. Then,

$$\alpha = \xi\, w_n = 0.5 \text{ x } 2.2 = 1.1$$

From diagram 8, the dominant roots are obtained which is the intersection of the $\xi = 0.5$ and $\alpha = 1.1$ curves. The system parameters that will yield the specified response are given by $T_f = 0.4$ sec., $T_m = 0.06$ sec., and $K = 1.9$. Similarly for other system requirements, the required system parameters can be easily obtained by the same procedure.

Comparison of Results with Root Locus Method

The validity of the proposed method can be verified by comparing the results with those obtained by the Root Locus method[1]. For T_f=0.25 sec., T_m = 0.625 sec. and ξ = 0.5, the Root Locus method[1] gives the following results:

overall gain $\quad K = 2.6$

dominant Poles $\quad s_{1,2} = -\alpha \pm j\beta = -1.6 \pm j2.8$

Referring to diagram 10, it is observed that for T_f = 0.25 sec., T_m = 0.0625 sec., ξ = 0.5, the following results are obtained:

overall gain $\quad K = 2.56$

$$\alpha = 1.6$$

$$\beta = 1.6 \tan 60^\circ$$

$$= 2.78$$

dominant Poles $\quad s_{1,2} = -\alpha \pm j\beta = -1.6 \pm j2.78$

Thus the new method not only gives results comparable to the Root Locus and Nyquist methods but it also yields information not obtainable from the other two methods.

CHAPTER IV

DISCUSSION AND CONCLUSION

A new method has been developed which can be effectively applied to the analysis and synthesis of linear feedback control systems. By mapping the stability boundary onto the imaginary axis of a new complex plane and realizing the coefficients of the transformed polynomial, the Hurwitz criterion can be applied to yield analytical expressions from which stability boundaries can be drawn for various values of system parameters. By using the system parameters as coordinates, the effect of simultaneous variation of system parameters on the dynamical behaviour of the system can be readily visualized. By locating the stable and unstable regions, relative stability analysis is reduced to the simple process of reading the diagrams. By superposing the α and ξ diagrams, the dominant poles can be located and system parameters are determined to give a desired response. Thus the new method provides a powerful tool for system analysis as well as design.

The proposed method is most general and can be applied to systems with any degree of complexity and under any specific kind of relative stability constraint. It is especially advantageous when applied to systems with two or more variable parameters.

APPENDIX

TABLE I

TABLE FOR COMPUTING THE COEFFICIENTS OF F(Z)

COEFFICIENT	IN TERMS OF COEFFICIENTS OF P(s)
A_{10}	a_5^2
A_9	$2a_5a_4\cos\theta$
A_8	$a_4^2 + 2a_5a_3\cos 2\theta$
A_7	$2a_4a_3\cos\theta + 2a_5a_2\cos 3\theta$
A_6	$a_3^2 + 2a_4a_2\cos 2\theta + 2a_5a_1\cos 4\theta$
A_5	$2a_3a_2\cos\theta + 2a_4a_1\cos 3\theta + 2a_5a_0\cos 5\theta$
A_4	$a_2^2 + 2a_3a_1\cos 2\theta + 2a_4a_0\cos 4\theta$
A_3	$2a_2a_1\cos\theta + 2a_3a_0\cos 3\theta$
A_2	$a_1^2 + 2a_2a_0\cos 2\theta$
A_1	$2a_1a_0\cos\theta$
A_0	a_0^2

where $P(s) = a_0 + a_1s + a_2s^2 + a_3s^3 + a_4s^4 + a_5s^5$

$$F(Z) = A_0 + A_1Z + A_2Z^2 + A_3Z^3 + A_4Z^4 + A_5Z^5 + A_6Z^6 + A_7Z^7 + A_8Z^8 + A_9Z^9 + A_{10}Z^{10}$$

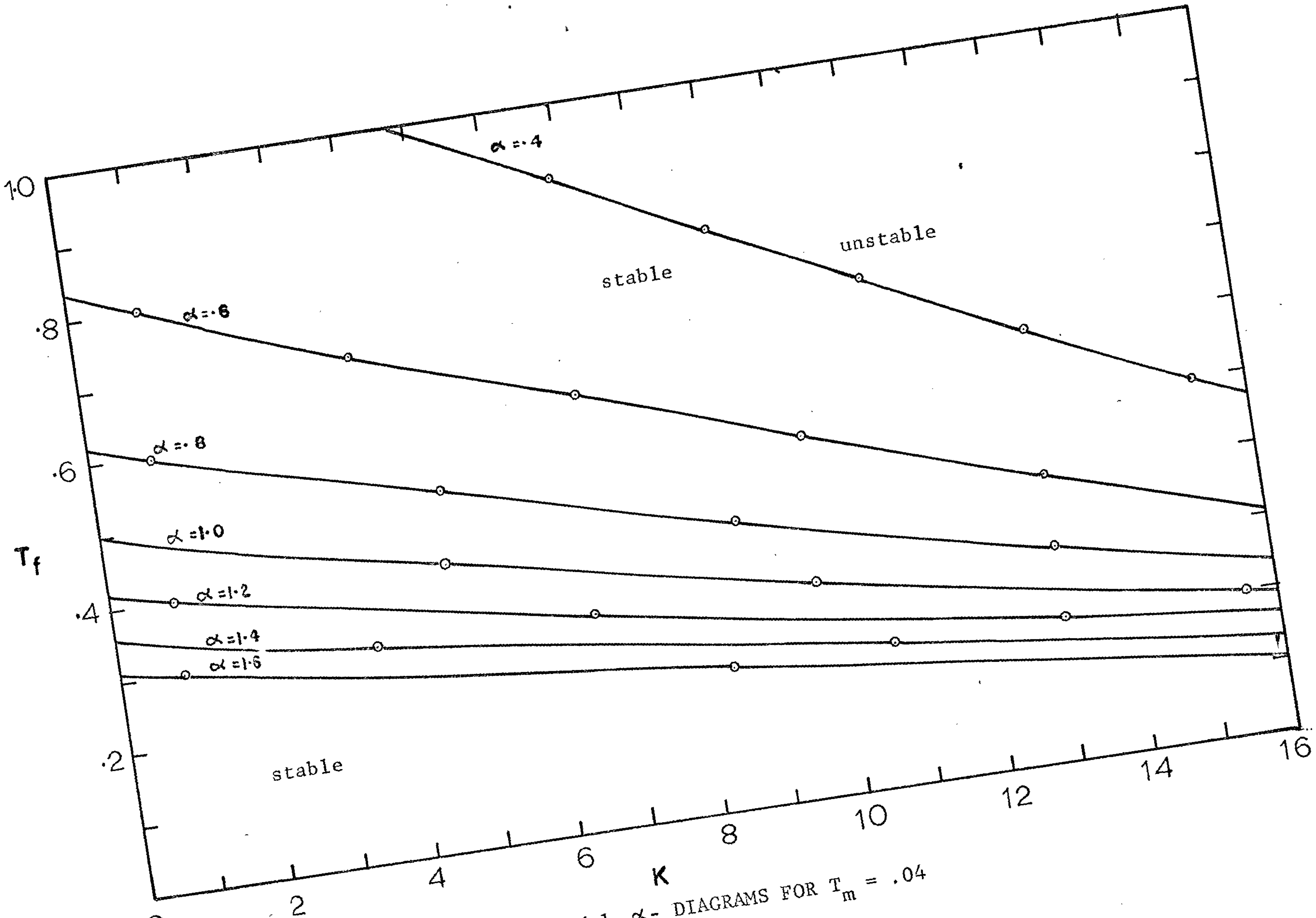

DIAGRAM 1 α- DIAGRAMS FOR T_m = .04

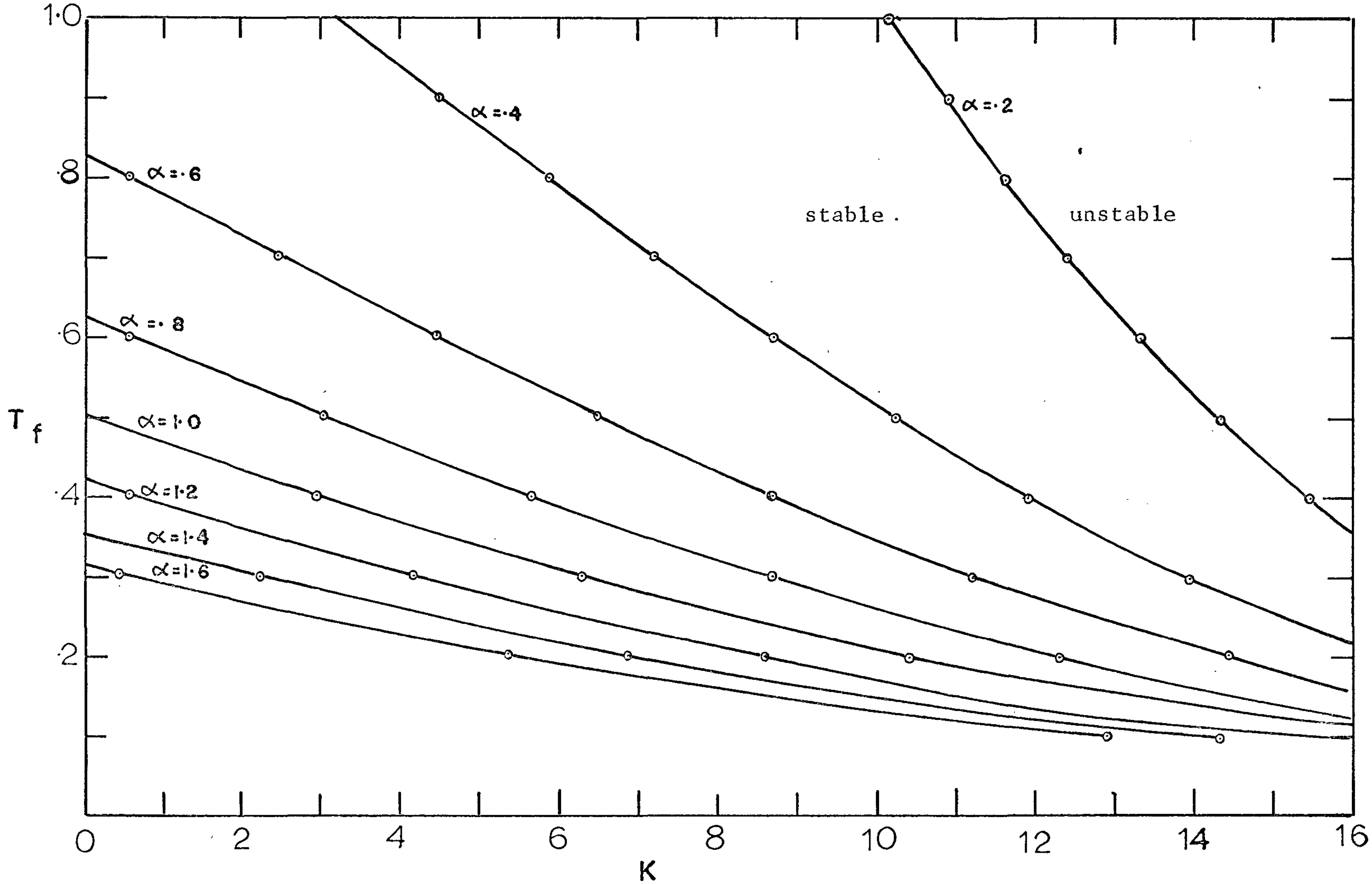

DIAGRAM 2. α - DIAGRAMS FOR T_m = .06

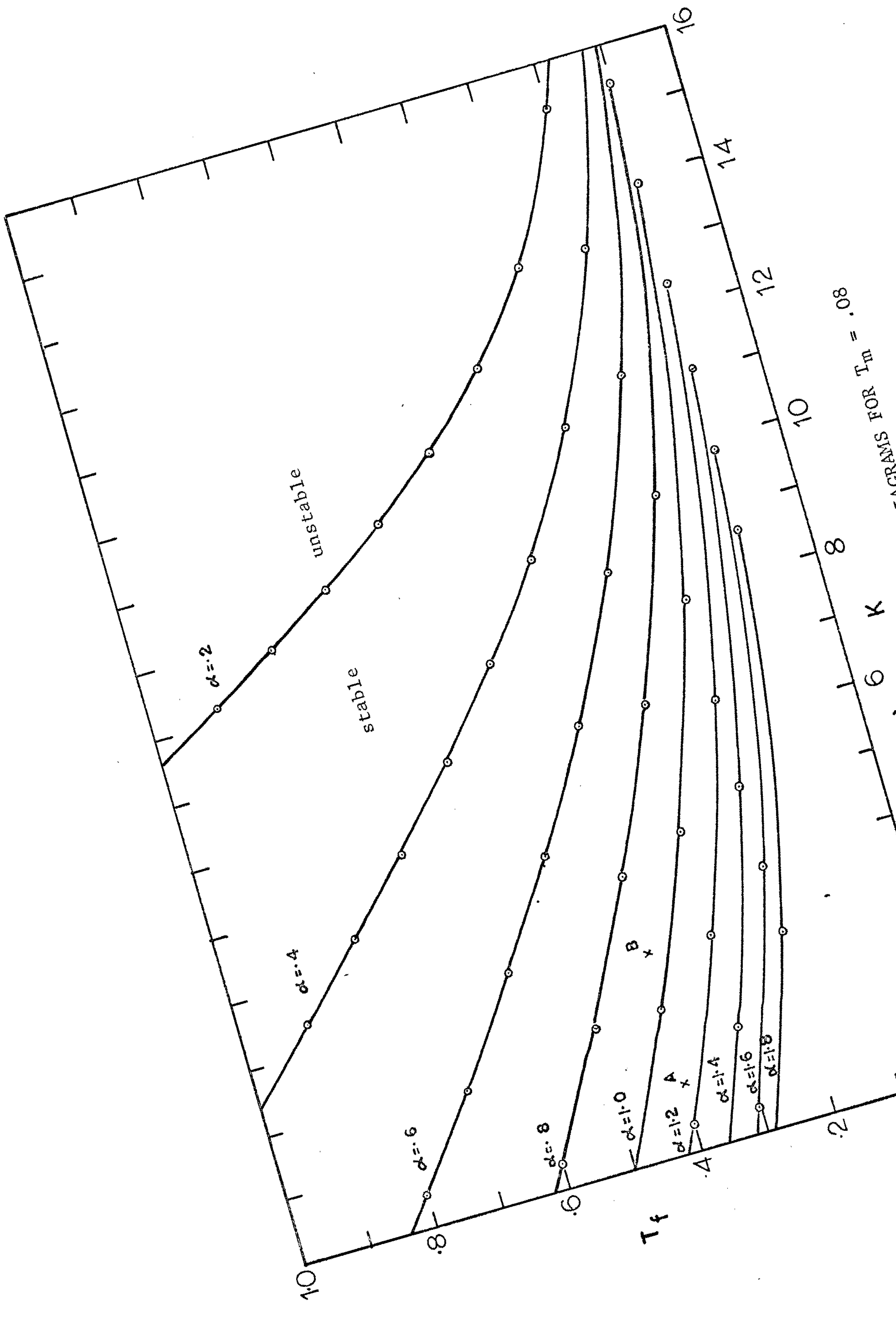

DIAGRAM 3. α - DIAGRAMS FOR T_m = .08

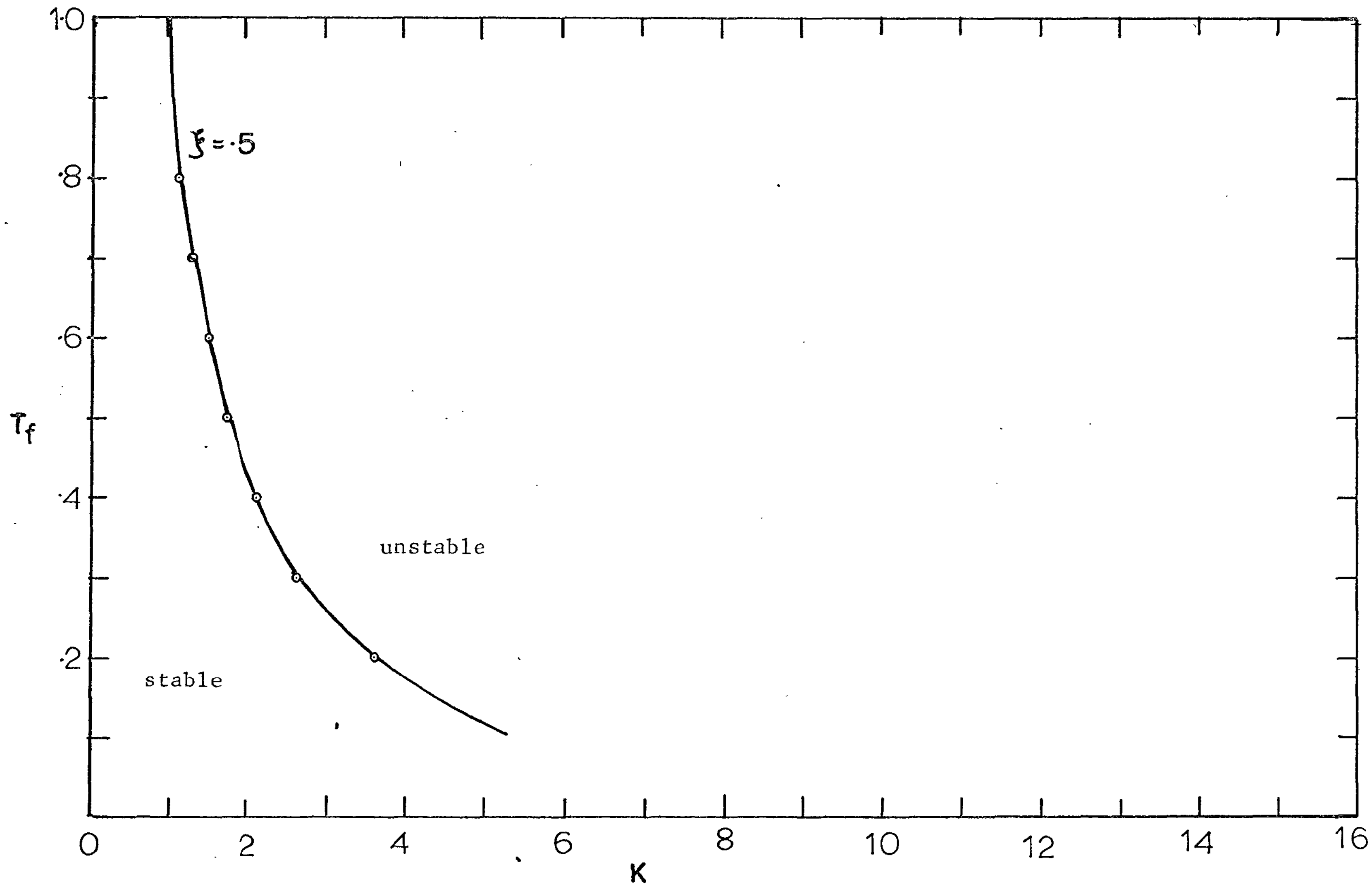

DIAGRAM 4. ξ - DIAGRAM FOR T_m = .04

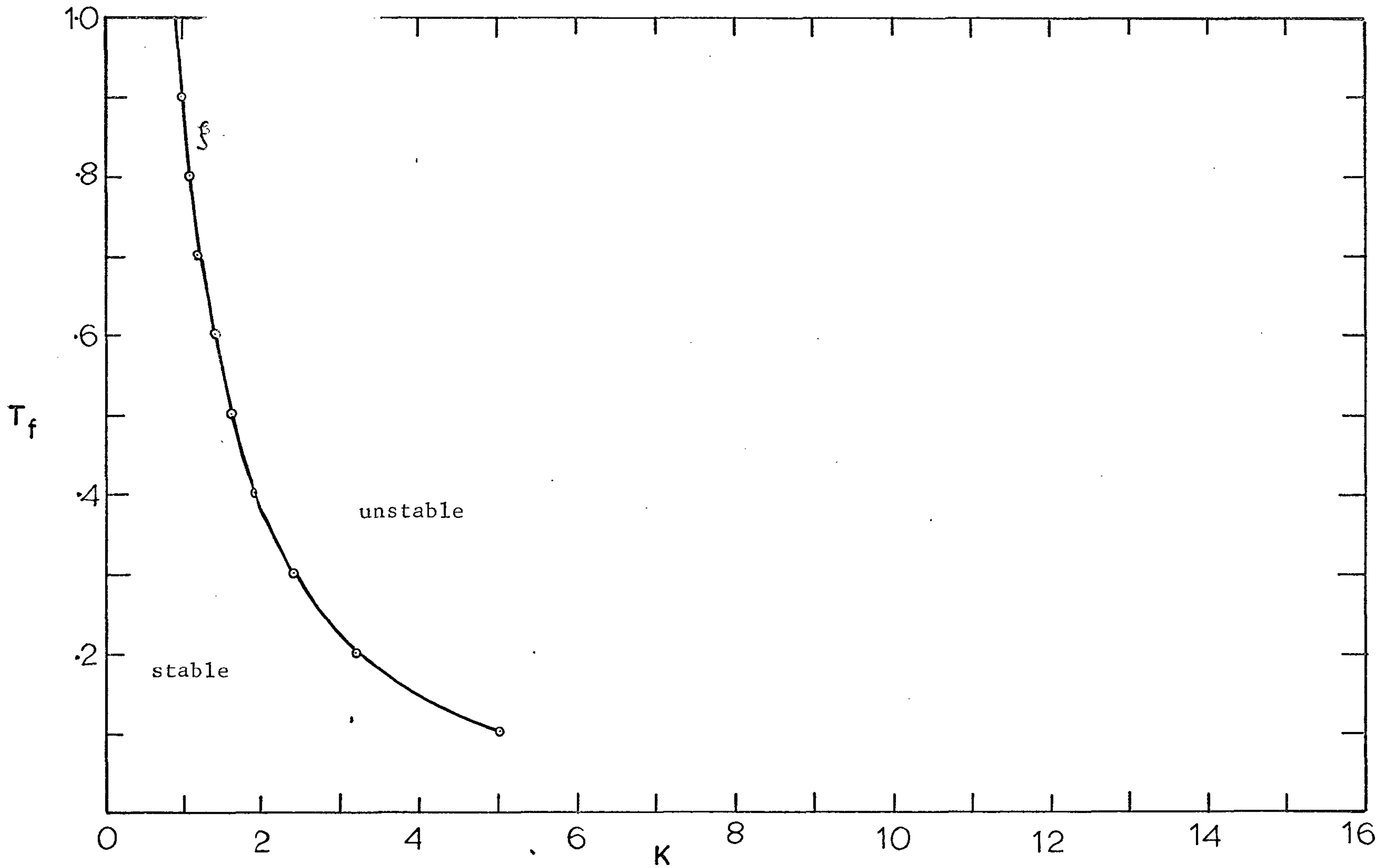

DIAGRAM 5. ξ - DIAGRAM FOR T_m = .06

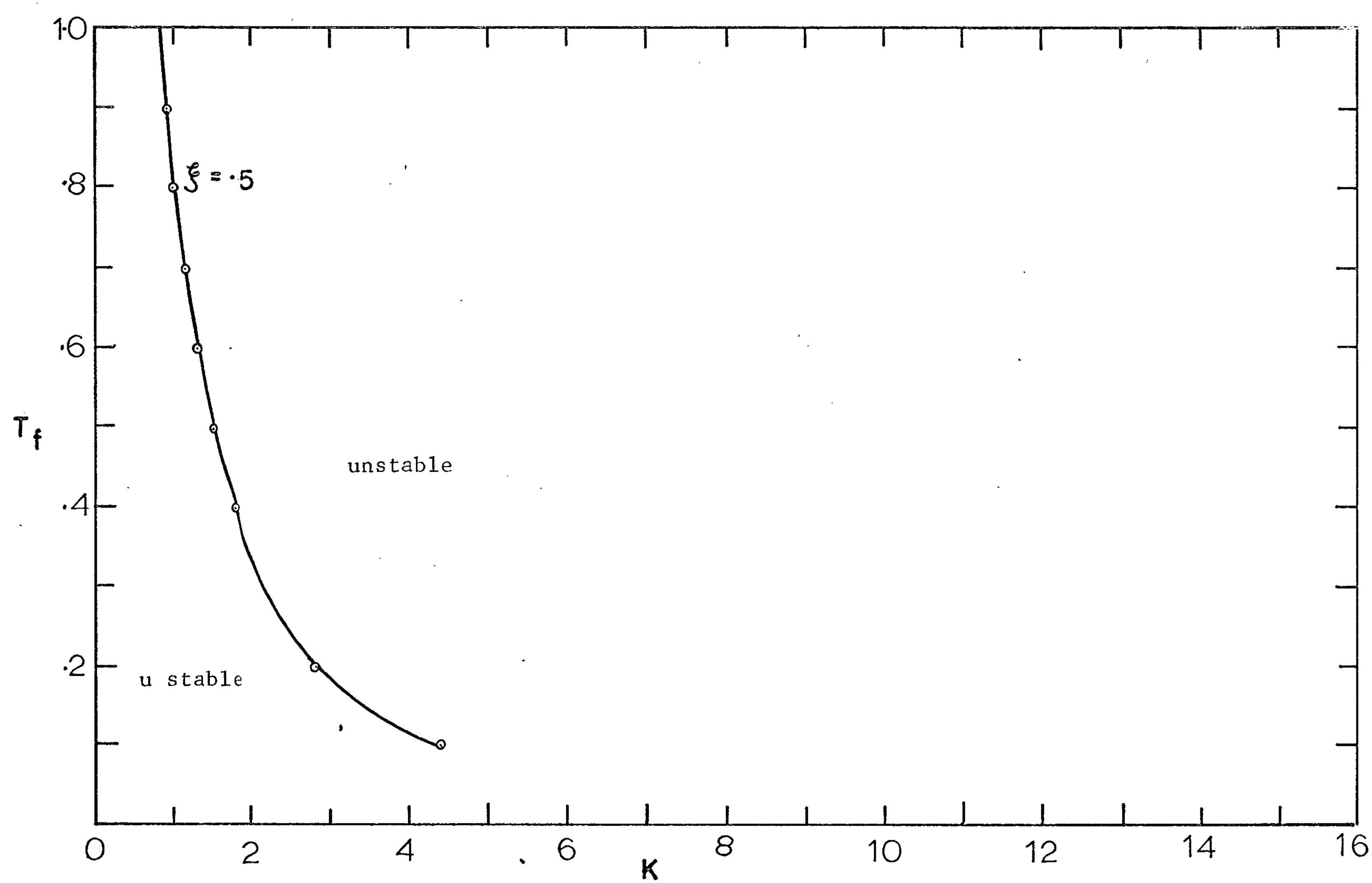

DIAGRAM 6. ξ - DIAGRAM FOR T_m = .08

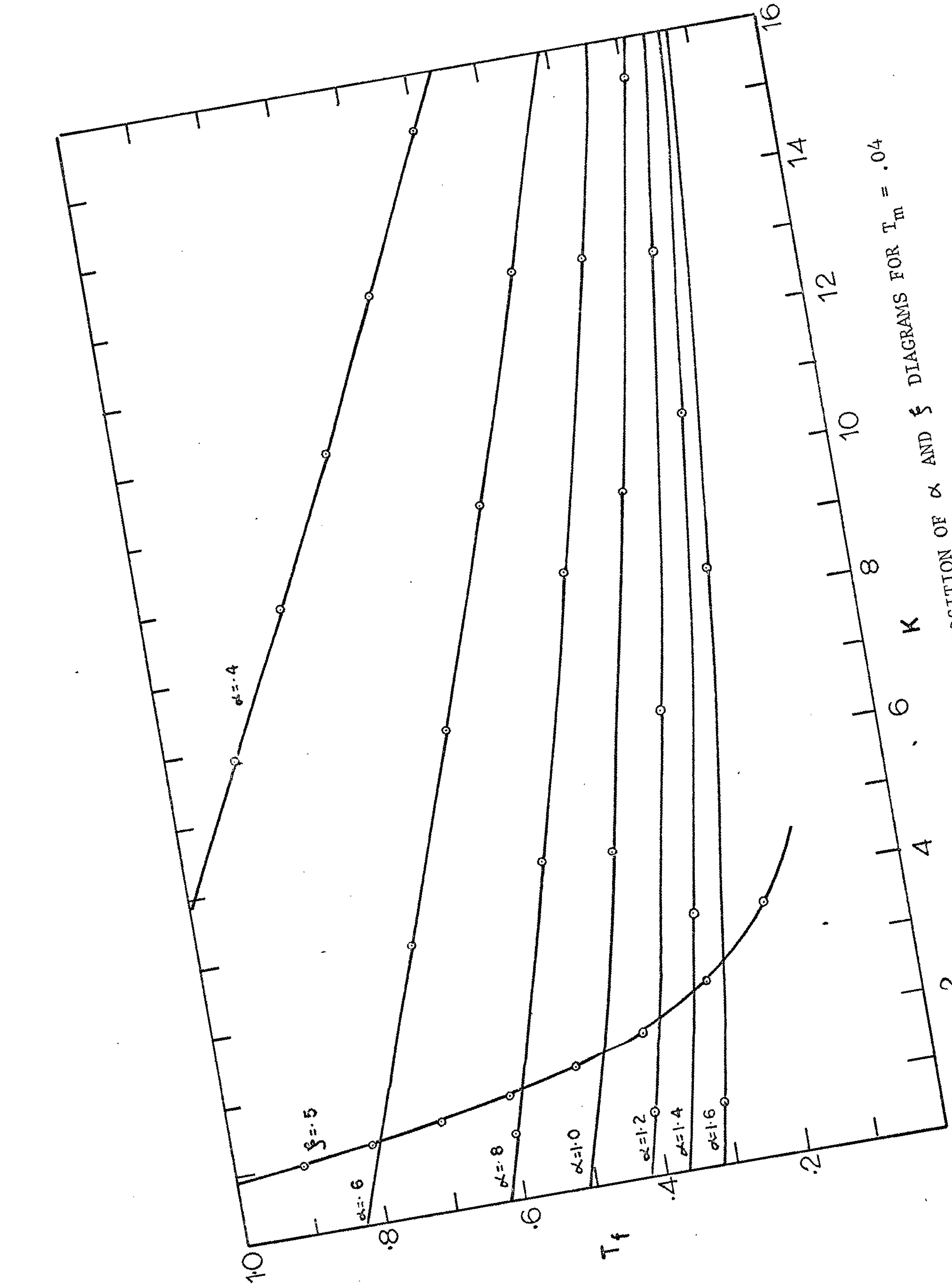

DIAGRAM 7. SUPERPOSITION OF α AND ξ DIAGRAMS FOR T_m = .04

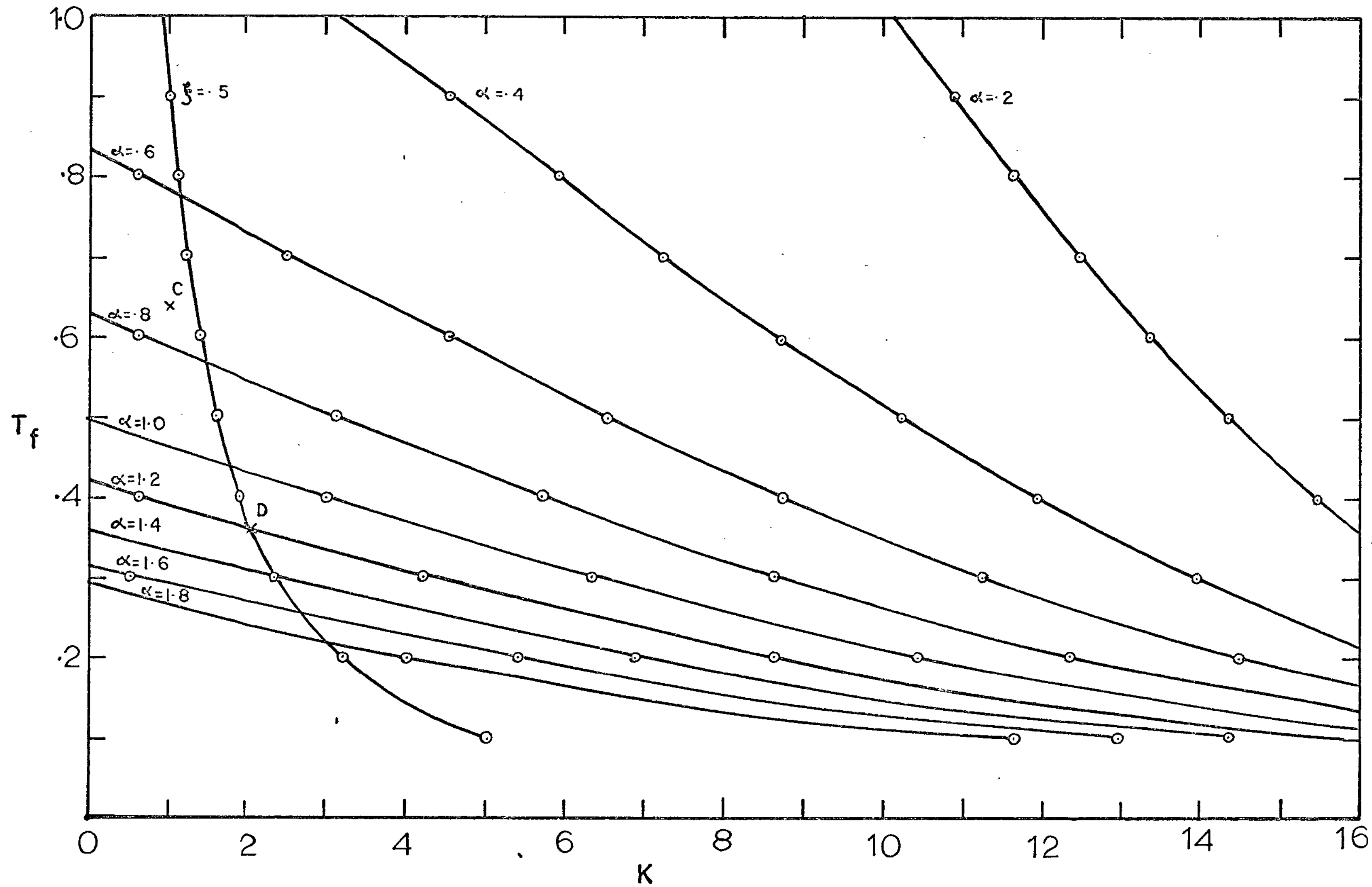

DIAGRAM 8. SUPERPOSITION OF α AND ξ DIAGRAMS FOR T_m = .06

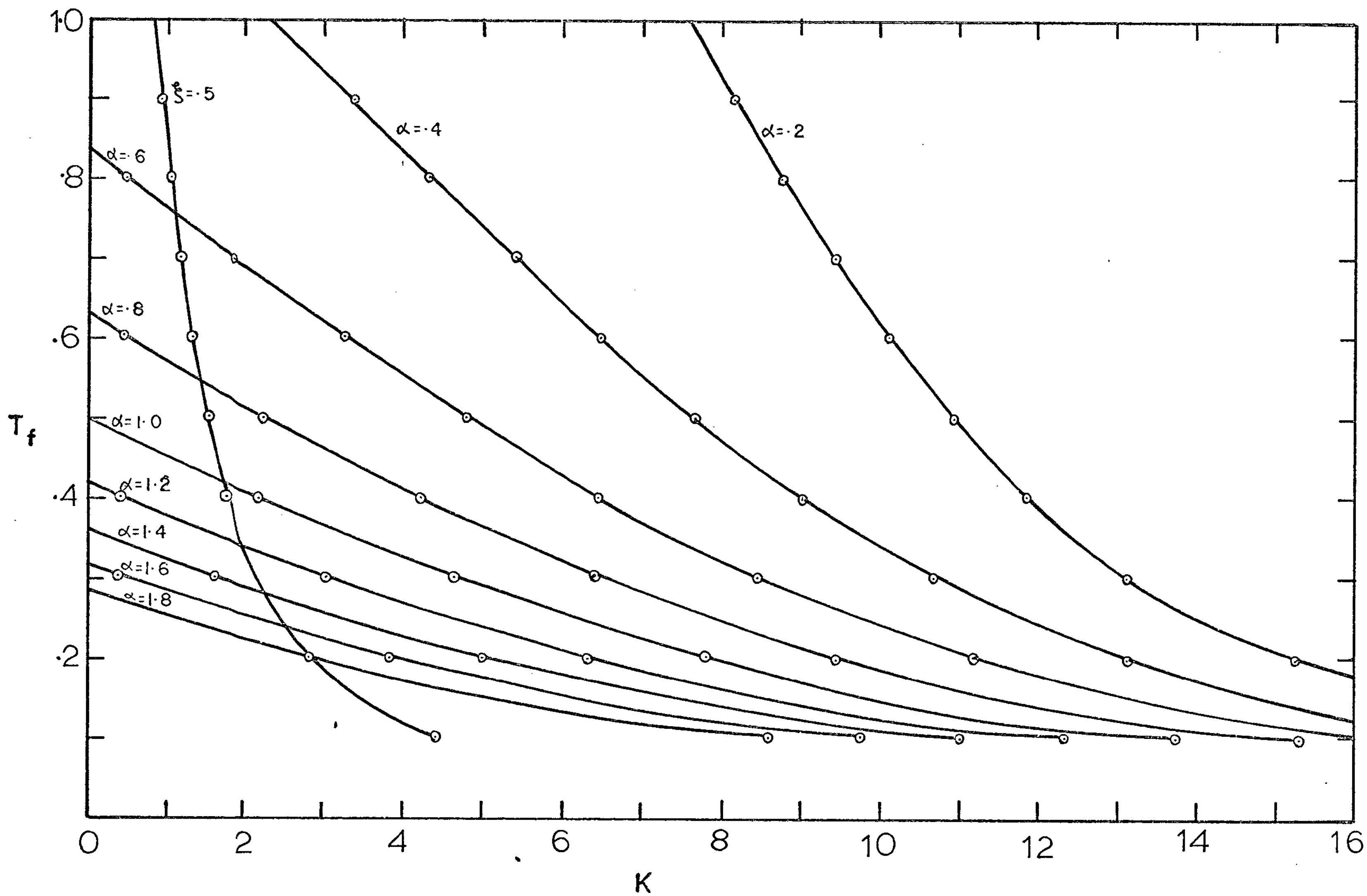

DIAGRAM 9. SUPERPOSITION OF α AND ξ DIAGRAMS FOR T_m = .08

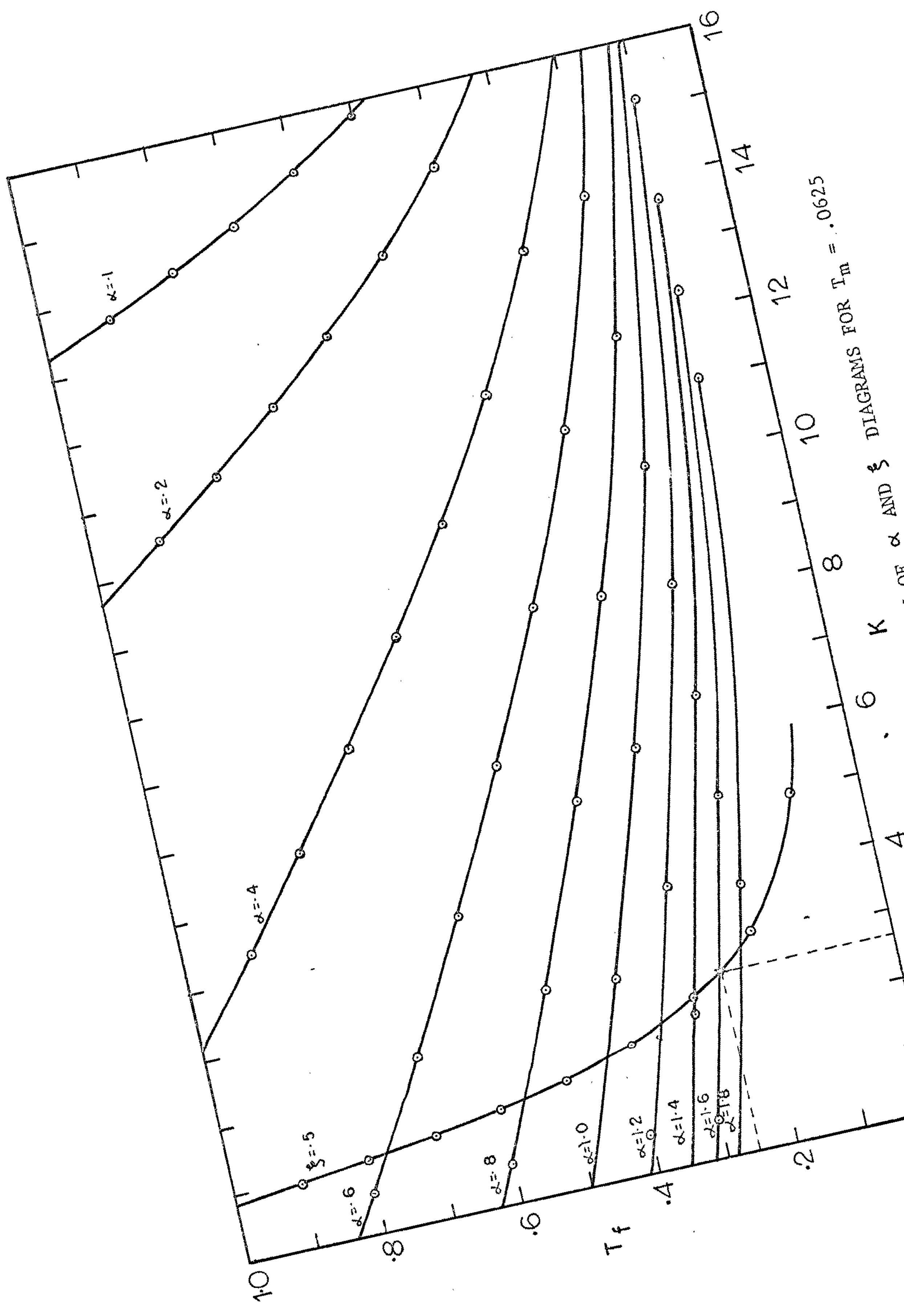

DIAGRAM 10. SUPERPOSITION OF α AND ξ DIAGRAMS FOR T_m = .0625

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